Learning Environment for Principles of Decomposition and Addition

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Principle of Decomposition

In mathematics just as in daily life we come across problems that are overwhelming because of their complexity or vast multiplicity. How can you approach such problems, in the battle to gain a good idea to conquer and master the task? Even the Romans knew how to win a battle: "divide et impera", which means "divide and reign". This is an old principle of warfare, already mentioned by the wise Chinese Sunzi 500 years before Christ in his book "The Art of War". Whereas it here refers to spreading disunity among the enemies, it has to be viewed differently in a mathematical context.

In short this heuristic principle, introduced in this unit, might be described by:

Please note:
If a problem is too overwhelming in its entirety or opportunity arises to decompose it into smaller parts, break it up and solve the big problem by solving its sub problems.

The scope of this principle is best understood, if we look at possible applications***:***

Example 1 Area Calculation

What size is the rectangle shown in figure 2?

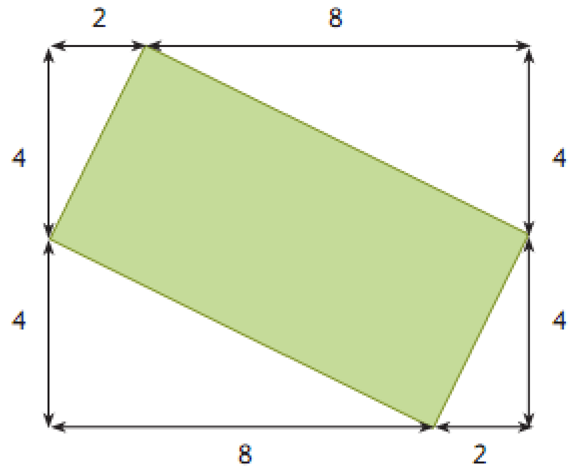

figure 2: decomposition and addition - rectangular 1

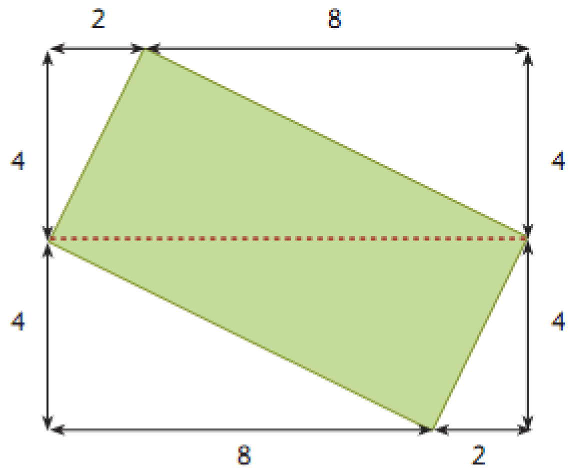

figure 3: decomposition and addition - rectangular 2

Solution process without decomposition:

The product of length and width gives the area of a rectangle. As symmetries and regularities of the figure are considered, it shows that they can be assessed by the Pythagorean theorem:

The width is: $\mathrm{B} = \sqrt{4^2 + 8^2} = \sqrt{80}$

and the length is: $\mathrm{L} = \sqrt{2^2 + 4^2} = \sqrt{20}$

So the area is: $\mathrm{A} = \mathrm{L} \cdot \mathrm{B} = 40$

Solution process with decomposition:

The application of the principle of decomposition is to break up the rectangular into two equal triangles, like outlined in the figure, allowing the calculation of the area by the data given. The dashed base has a total width of 8 + 2 = 10, and the heights is 4 in each case. So the result can be concluded directly as: $A = 2 \cdot \frac{1}{2} \cdot 4 \cdot 10 = 40$

This geometric example demonstrates the advantages and the idea of the principle of decomposition. The first solution process (without decomposition) demonstrates the attempt to trace back the area calculation of the rectangle to a familiar formula. The advantage of the second option (with decomposition) is that it is a more direct way without detours over geometric propositions. No radical or Pythagorean theorem is required, but simply the formula of assessing the area of a triangular.

Decompositions of this kind usually rely on regularities or symmetries within the problem - like in the example above in the given rectangle. Therefore the principle of decomposition and the principle of symmetry show a strong link. Such decompositions are utilized in daily life in assessing the areas of an apartment.

Problem 1: Area calculation of a room with a complicated layout

Figure 4 shows a room with a complicated layout (measures are given in meters). Paul wants to lay a carpet in that room. How many square meters does he have to buy? Assess the area by means of the principle of decomposition! [solution: 54m²]

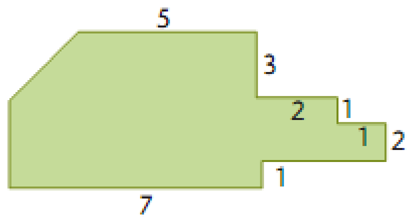

figure 4: decomposition and addition – layout

This principle is not just applied with area assessments; the inner angle theorem can be accomplished too.

Example 2 Angle sum principle

What is the sum of the inner angles of the hexagon?

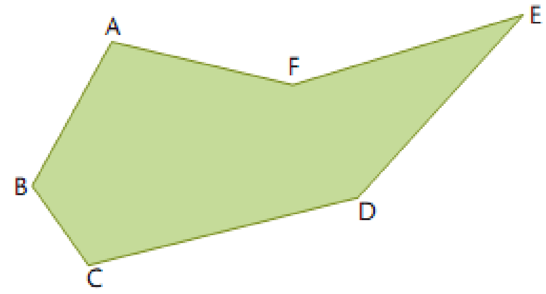

figure 5: decomposition and addition – hexagon

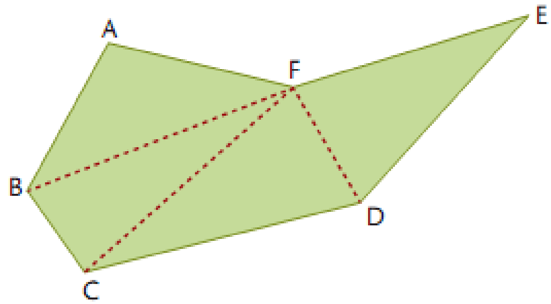

figure 6: decomposition and addition – triangles

Solution:

Break up the hexagon like sketched in figure 6 into four triangles. By means of this decomposition the angle sum of the hexagon can be calculated from the angle sums of the four triangles, which should always make up 180°. So the inner angle sum of the hexagon is $4 \cdot 180° = 720°$.

Problem 2: Angle sum for an n-gon

Generalize the correlation of the angle sum of the hexagon for any n-gon.

What has been demonstrated on two-dimensional objects can also be applied to three-dimensional bodies. A dome`s volume for example, could approximately be split up into a cylinder and a hemisphere.

Problem 3: Composited body

Draw a body that can be decomposed into various elemental known bodies and assess its volume. Present the problem to your class mates!

In geometry this principle can be illustrated really well, but it is applicable far beyond just that field. Even when first getting to know mathematics, as the basic arithmetic operations are acquired, the decomposition principle is used, as is called "intelligent calculation".

Example 3 Intelligent Calculation

Calculate by heart the sum of: 8984 + 797.

Solution:

To calculate skillfully, students are taught in primary school to decompose the sum in their mind:

$8984 + 787 = 8984 + 16 + 771 = 9000 + 771 = 9771$

Even with more complicated calculations a systematic and purposeful decomposition often leads to an aim. For example with assessing a quantity of divisors, where possible factors are more or less tried out or using the direct option of prime factors of all possible factors.

Example 4 Assessing a quantity of divisors

Calculate all divisors of the number 117.

Solution process without decomposition:

You work through all numbers from 1 to 117 and test if they are divisors of 117. By trying this the amount of factors can be assessed. However, it would solve the problem much faster if you would first skip all factors that are non-dividers and their multiple.

Solution process with decomposition:

Prime decomposition of $117 = 3 \cdot 3 \cdot 13$ is assessed first. The divisors are all products of those prime factors, $T = \{1,3,9,13,39,117\}$.

The procedure to assess zeroing of functions is similar to this.

Problem 4: Zeroing of a function

Assess all zeroing of the equation:

$f(x) = (x^2 + 4x + 4) \cdot (x - 2) \cdot (x - 3)^2.$

How can the principle of decomposition be utilized here?

Hint:

Trying all different ways is pretty challenging. But the various ways can be separated, because there are points that the way has to go through for sure.

Even with algorithms the principle of decomposition is applied i.e. in the context of sorting or optimization. If you wanted to know the shortest distance between two places over several stopovers, the problem is modeled into a picture of networks and points. Those points could be cities and the edges connecting them streets of a certain length. When looking for the shortest distance in minor problems, you could reach a result pretty quickly by systematic testing. An ingenious decomposition however could speed up that process tremendously.

Problem 5: Looking for the shortest distance

Find the shortest distance from point A to point R. The numbers on the edges state the length of each way. One way has the length of the sum of the edges that it runs over.

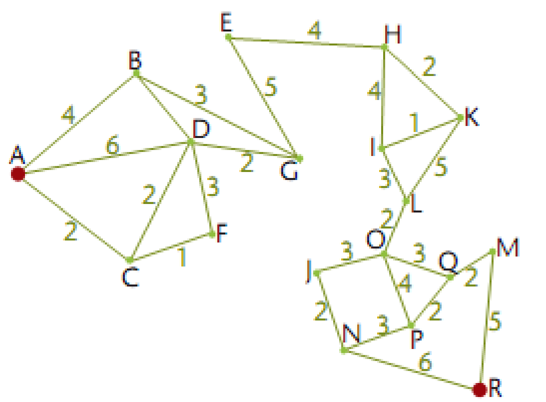

figure 7: decomposition & addition - shortest distance

The above sample utilized the feature that an optimal way from point A to point R consists only of optimal partial sections. Which means: If the optimal way runs over the points G and N, there is no other shorter way from G to N than the one covered in the optimal way. If such a way would exist, this part could be replaced within the optimal way found, leading up to a better optimal way, which of course could not be, because the optimal way has ultimately been optimal already. The so called Dynamic Programming benefits rom this principle, to develop procedures to solve the problem of the shortest distance on a general basis.

In high school you will become acquainted withmore contexts where the principle of decomposition is applied. Calculating the derivative of a function for example, which is given by the sum of several terms, is decomposed according to the sum rule in the sum of functions and those are to be derived. The same is done with integration. In both cases the linearity of each operation is taken advantage of, being a very general and useful concept of mathematics, promoting the decomposition principle in many coherences.

The decomposition principle is an essential method of mathematics, especially applied in proving propositions. Usually case differentiations are done in mathematics when something needs to be proven. The general situation is divided into different special partial problems that cover all possible cases.

The ability to analyze a situation and to decompose a problem into certain components is a characteristic skill of the human mind. In all sciences procedures are analyzed by that kind of decomposition. Much of our understanding of procedures in the world relies on the ability to decompose things so we can comprehend and grasp them. It is worth to be aware of and to internalize the practice of decomposition.

Principle of Addition

A principle closely related to decomposition is the principle of addition. The addition principle is less intuitive than the principle of decomposition, because it contradicts the intuitive mental equivalence of "simplifying" and "decomposing". A problem is supplemented so it becomes more extensive, but can be traced back to something familiar. The principle of addition will be introduced by reference to a geometric proof - probably remembered from school, the Pythagorean theorem.

Example 5 Proof of Pythagorean`s theorem
The sum of the areas of the two squares on the legs equals the area of the square on the hypotenuse.

Solution process with the help of the principle of addition:
By just looking at the triangle with the corners a,b,c; it seems difficult at first to prove this proposition. The problem can be visualized by adding square areas above the edges of the triangle:

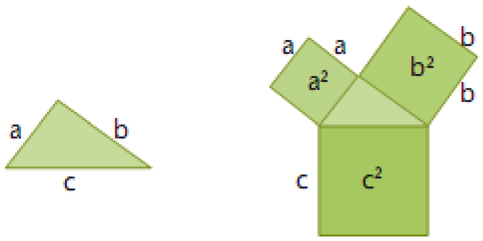

figure 8: decomposition & addition - pythagorean theorem 1

Now the pythagorean theorem can be proven by appropriate order of the square areas in combination with the given triangle by further additions:

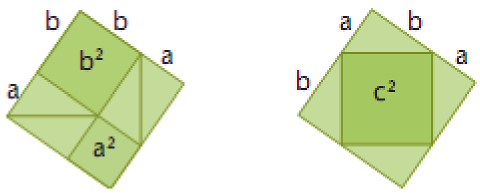

figure 9: decomposition & addition - pythagorean theorem 2

As clearly seen in these two figures, the sides of the big squares each have a length of a + b and both big squares always contain four of the original triangles. As a consequence the square above the hypotenuse must be the same size as the two other squares above the legs. This proof is also perfect to show why this proposition is only valid for rectangular triangles.

The principle of addition is not just often used in geometry. A very important procedure of solving quadratic equations is based on the same principle: The procedure of quadratic addition.

Example 6 Quadratic Addition

Look at the equation: $x^2 + 6x = 27$.

It is probably hard to "see" a solution here. Even the principle of decomposition does not help. But the equation can be supplemented by adding the number 9 to both sides, so binomial formula can be applied:

$x^2 + 6x + 9 = 27 + 9 \rightarrow (x+3)^2 = 9$

$x + 3 = \pm 6 \qquad \rightarrow x = 3 \text{ or } x = -9$

Both solutions were found.

Another typical example for the application of the principle of addition are proofs of inequalities. Usually something is added on the bigger side, to trace back the term to a familiar expression. First is: $x^2 + 4x + 4 = (x+2)^2 \geq 0$. We can add 1 on the left side. By that we probably reduce it, which results directly in:

$$
\begin{aligned}
x^2 + 4x + 5 &\geq 0 \\
x^2 + 4x + 5 &= x^2 + 4x + 5 + (1-1) \\
&= x^2 + 4x + 4 + 1 \\
&\geq x^2 + 4x + 4 = (x+2)^2 \\
&\geq 0
\end{aligned}
$$

Similar to the principle of decomposition the following basic idea may be recorded:

Please note:
Look for suitable additions /supplements of a problem to transform it into a known problem (→ principle of analogy), or

- Look for suitable additions /supplements of a problem to create symmetries (→ principle of symmetry)

The question is: How do you come up with suitable additions? Contrary to the principle of decomposition, where there is a reasonable amount of useful possibilities, there are usually endless additions possible. How can useful additions be recognized? For this it is important to know problem context related regularities. If the goal is something like a proof, regularities are searched for to reach that goal, trying to extend the problem, so these regularities can be applied. If the aim is unknown, maybe because it is an open problem, some trying may be necessary like shown in the following exercise.

Problem 6: Defining the relation between angles

By means of the principle of addition the following figure can be extended so that a geometric relation between the angles A, B and C can be shown. Which relation is there between the angles?

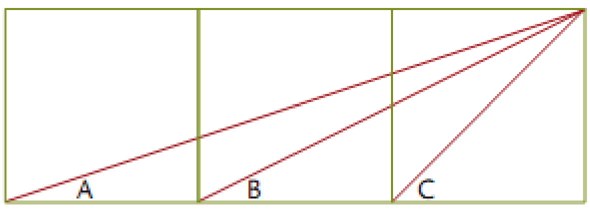

figure 10: decomposition & addition – angles

Problem 7: Altitudes

Proof similar to the proof presented with the Pythagorean theorem, that the square of the heights is the same as $p \cdot q$ in a rectangular triangle.

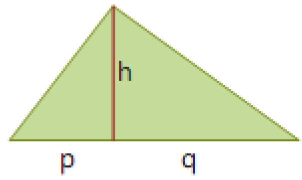

figure 11: decomposition & addition – altitudes

Hints for a solution:

It would help to clarify how connected it is by trying some values. Even without a specific goal or knowledge of the context, it is possible to come up with some ideas of useful additions: To get a geometric connection between three angles, it seems a good idea to extend the figure, so we obtain a geometric figure, containing all three angles or just combinations of those three angles. If you cannot come up with ideas yourself here are a few hints:

Hint 1:

Rather than just extending the three given triangles, you can also add them in a new arrangement: rotated, mirrored or replaced.

Hint 2:

The regularity is: A + B = C

Hint 3:

Try to construct a new, bigger triangle with all angles so a formula can be developed.

Here are two hints, if you don`t know how to proceed.

Hint 1:

Decompose the triangle into two part triangles, containing p and h as well as h and q like sketched in the above figure.

Hint 2:

Add these two part triangles by the square area of the heights and then by the rectangular area $p \cdot q$.

With the last problem the principle of decomposition and the principle of addition were applied. Do you see where?

Often a combination of the principles of decomposition and addition prove to be useful. In doing so a problem is decomposed, but also usefully supplemented. To illustrate the advantage of this procedure some more, another example will be presented here. We will look at the derivation of the area formula of a parallelogram, saying: A = base · heights.

Example 7 Area formula of a parallelogram

First we decompose the following parallelogram:

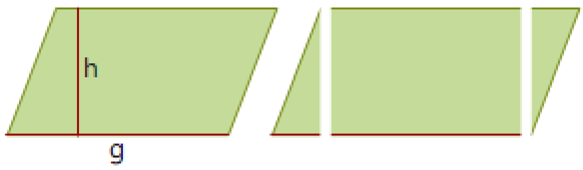

figure 12: decomposing & addition – parallelogram 1

Pure decomposition does not get us anywhere, because we only don`t know the length of the two red distances in the right figure, we only know the length of the complete base line g in the left figure. So we supplement the figure by "gluing" the left triangle on the right side. By doing so we get a rectangle with the edge lengths g and h like shown in the next figure:

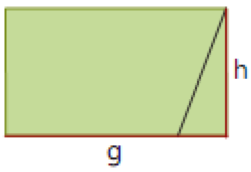

figure 13: decomposition & addition - parallelogram 2

Conclusion

The principles of decomposition and addition are closely related and based on the same idea: to transform a problem, so the known can be recognized. The principle of decomposition can be applied more intuitively and is easier to deploy since possible solutions are manageable. Both principles cannot be exchanged. They rather supplement each other, therefore knowing both principles is useful.

Both principles are part of daily life just like mathematics for example in the way our eyes process pictures. Usually our brain decomposes or supplements structures to recognize known things. In the picture on the bottom right for example our brain adds "white edges", so a triangle appears even though those edges don`t exist.

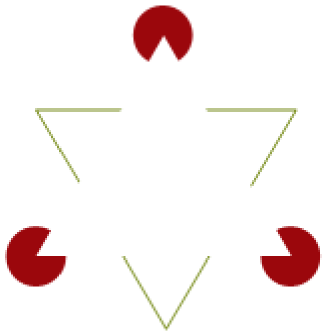

figure 14: decomposition & addition - optical illusion